Int. J. Solids Structures, 1976, Vol. 12, pp. 559–575. Pergamon Press. Printed in Great Britain

FINITE AND SYMMETRIC THERMOMECHANICAL WAVES IN MATERIALS WITH INTERNAL STATE VARIABLES

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(*Received* 15 *September* 1975; *revised* 23 *January* 1976)

Abstract—The paper is devoted to the study of wave propagation through materials with internal state variables under two main assumptions: finite speed of propagation and symmetry of acceleration waves. Additional constitutive assumptions are also used: heat flux does not depend on the temperature gradient and the time-derivatives of internal state variables are linear functions of the temperature gradient. Under all these hypotheses, in the neighborhood of a strong equilibrium state, one finds four real and symmetric possible acceleration waves, at least two of them being coupled waves, and heat flux results an internal state variable. All these results are obtained in the general three-dimensional case. As an illustration, the isotropic linear theory is considered, where both acceleration and shock waves are treated.

INTRODUCTION

Wave propagation through materials for which the heat flux is determined by a constitutive equation of Maxwell–Cattaneo type, has been studied by several authors, from different points of view. In the frame-work of non-linear theories one generally finds that acceleration waves are not symmetric with respect to the propagation direction unless additional hypotheses are laid down. (See Gurtin and Pipkin [3], Kosiński and Perzyna [2]; for additional comments see Suliciu [4].)

The authors of the present work do consider that symmetry of acceleration waves is a requirement of a physical nature. This condition has been investigated by Suliciu [4] in the one-dimensional case, for materials with internal state variables but using particular constitutive assumptions that give the constitutive equation of heat flux a form which is close to Cattaneo's equation [6, 7]. The symmetry condition as well as certain informations on the equilibrium state imply the existence of real acceleration waves in a neighborhood of an equilibrium state. Using the same assumptions one proves that heat flux depends only on the internal state variables and therefore, it has no jump across a shock wave; moreover, in the neighborhood of an equilibrium state, the shock wave speed is close to the adiabatic sound speed.

Using the additional assumption that the equilibrium state is a strong equilibrium state in the sense of Truesdell [8], Suliciu [5] finds similar results in the general one-dimensional theory of materials with internal state variables.

The present work generalizes the above results to the general three-dimensional theory of wave propagation through materials with internal state variables. The model used for such a material is that of Coleman and Gurtin [1] with certain additional constitutive assumptions. Moreover, internal state variables are chosen to be frame indifferent but at least a certain number of these variables change under unimodular mappings of the reference configuration. Therefore, internal state variables will somehow have a broader meaning here than that used by Coleman and Gurtin [1] (see also Bowen [9], Bowen and Wang [10], Bowen and Chen [11]).

The hypothesis that heat flux does not explicitely depend on the temperature gradient brings this model close to that of a material with fading memory due to Gurtin and Pipkin [3], and to the Maxwell-Catteneo constitutive assumption. According to this hypothesis and to the assumption that the evolution function of the internal state variables is linear in the temperature gradient, the system of equations that describes thermodynamic processes in the body becomes quasilinear. Two symmetry assumptions are laid down, concerning the wave speeds (Section 2) and the mechanical amplitude of the wave (Section 2), for bodies with non-vanishing thermal dilatation at a strong equilibrium state (condition (4.7)). The strong ellipticity condition together with the above hypotheses lead to the conclusion that in the neighborhood of a strong equilibrium state there are four real and symmetric possible acceleration waves, and at least two of them carry jumps of the temperature derivatives. At a strong equilibrium state, the heat flux as well as its derivatives with respect to temperature and deformation gradient vanish. Therefore, the propagation speeds of small amplitude shocks propagating over a strong equilibrium state are close to the adiabatic sound speeds corresponding to that strong equilibrium state.

The isotropic linear theory is treated in the last two sections, in order to illustrate the above mentioned results. One can see that heat flux becomes an internal (vector) state variable and therefore, the Cattaneo constitutive equation becomes an evolution equation for this internal state variable.

1. PRELIMINARIES

The notations we use in this work are mostly those of Coleman and Gurtin [1]. For an easier reading, we reproduce them briefly.

The motion of a body $\mathcal{B}$ with respect to an initial reference configuration $\mathcal{R}$ is described by the function $\mathbf{x} = \chi(\mathbf{X}, t)$, which is a one-to-one correspondence from $\mathbf{X}$ to $\mathbf{x}$, where $\mathbf{X} \in \mathcal{R}$, $\mathbf{x}$ gives the coordinates in the actual configuration and t is time. For simplicity one considers only cartesian coordinates for $\mathbf{X}$ and $\mathbf{x}$.

The quantities

$$\mathbf{F} = \text{Grad}\,\chi = \left(\frac{\partial \chi_i}{\partial X_j}\right), \quad \mathbf{v} = \dot{\mathbf{x}} = \left(\frac{\partial \chi_i}{\partial t}\right) \tag{1.1}$$

are called the deformation gradient and the particle velocity respectively. One usually assumes that $\det \mathbf{F} = J > 0$.

We also use the following notations: $\mathbf{T} = (T_{ij})$ is the symmetric Cauchy stress tensor, $\tilde{\mathbf{S}}$ and $\mathbf{S}$ given by

$$\tilde{\mathbf{S}} = J\mathbf{T}(\mathbf{F}^{-1})^T = \rho_0 \mathbf{S} \tag{1.2}$$

are the Piola–Kirchoff stress tensors, ρ_0 is mass density in the reference configuration, e is the specific internal energy, η is specific entropy, θ is temperature, $\theta > 0$, ψ defined as

$$\psi = e - \theta\eta \tag{1.3}$$

is the specific free energy,

$$\mathbf{g} = \left(\frac{\partial \theta}{\partial x_i}\right) = \text{grad}\,\theta \tag{1.4}$$

is the temperature gradient, $\mathbf{q}$ is the heat flux, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N)$ is the vector of internal state variables and ρ is actual mass density.

Balance of mass is expressed by

$$\rho_0 = \rho J. \tag{1.5}$$

Balance of momentum and energy are given by

$$\begin{gathered} \text{Div}\,\tilde{\mathbf{S}} + \rho_0 \mathbf{b} = \rho_0 \ddot{\mathbf{x}}, \quad \frac{\partial \tilde{S}_{ij}}{\partial X_j} + \rho_0 b_i = \rho_0 \frac{\partial v_i}{\partial t} \\ \rho_0 \dot{\psi} + \rho_0(\dot{\theta}\eta + \theta\dot{\eta}) - \tilde{\mathbf{S}} \cdot \dot{\mathbf{F}} + \text{Div}\,\tilde{\mathbf{q}} = \rho_0 r \end{gathered} \tag{1.6}$$

where b are body forces, r is the heat supply and

$$\begin{gathered} \tilde{\mathbf{S}} \cdot \dot{\mathbf{F}} = \tilde{S}_{ij} \dot{F}_{ij} \\ \tilde{\mathbf{q}} = J\mathbf{F}^{-1}\mathbf{q}, \quad \mathbf{g}_0 = \text{Grad}\,\theta = \left(\frac{\partial \theta}{\partial X_i}\right) = \mathbf{F}^T \mathbf{g}. \end{gathered} \tag{1.7}$$

The second law of thermodynamics is here expressed by Clausius–Duhem inequality

$$-\dot{\psi}-\eta\dot{\theta}+\frac{1}{\rho_0}\tilde{\mathbf{S}}\cdot\dot{\mathbf{F}}-\frac{1}{\rho_0\theta}\tilde{\mathbf{q}}\cdot\mathbf{g}_0\geq 0. \tag{1.8}$$

The constitutive assumptions are

$$\begin{aligned}\psi&=\hat{\psi}(\mathbf{F},\theta,\mathbf{g},\boldsymbol{\alpha})\\ \eta&=\hat{\eta}(\mathbf{F},\theta,\mathbf{g},\boldsymbol{\alpha})\\ \tilde{\mathbf{S}}&=\hat{\mathbf{S}}(\mathbf{F},\theta,\mathbf{g},\boldsymbol{\alpha})\\ \tilde{\mathbf{q}}&=\hat{\mathbf{q}}(\mathbf{F},\theta,\mathbf{g},\boldsymbol{\alpha})\\ \dot{\boldsymbol{\alpha}}&=\mathbf{f}(\mathbf{F},\theta,\mathbf{g},\boldsymbol{\alpha})\end{aligned} \tag{1.9}$$

where all functions $\hat{\psi}$, $\hat{\eta}$, $\hat{\mathbf{S}}$, $\hat{\mathbf{q}}$ and $\mathbf{f}$ are as smooth as required by further calculations.

Inequality (1.8) imposes the following restrictions on the constitutive equations (1.9) (Coleman and Gurtin[1]):

$$\begin{aligned}\psi&=\hat{\psi}(\mathbf{F},\theta,\boldsymbol{\alpha})\\ \eta&=\hat{\eta}(\mathbf{F},\theta,\boldsymbol{\alpha})=-\frac{\partial\hat{\psi}}{\partial\theta}(\mathbf{F},\theta,\boldsymbol{\alpha})\\ \tilde{\mathbf{S}}&=\hat{\mathbf{S}}(\mathbf{F},\theta,\boldsymbol{\alpha})=\rho_0\frac{\partial\hat{\psi}}{\partial\mathbf{F}}(\mathbf{F},\theta,\boldsymbol{\alpha}).\end{aligned} \tag{1.10}$$

We now make two additional constitutive assumptions

$$\begin{aligned}&\frac{\partial\hat{\mathbf{q}}}{\partial\mathbf{g}_0}(\mathbf{F},\theta,\mathbf{g}_0,\boldsymbol{\alpha})=\mathbf{0}\\ &\mathbf{f}(\mathbf{F},\theta,\mathbf{g}_0,\boldsymbol{\alpha})=\mathbf{A}(\mathbf{F},\theta,\boldsymbol{\alpha})\mathbf{g}_0+\mathbf{b}(\mathbf{F},\theta,\boldsymbol{\alpha})\end{aligned} \tag{1.11}$$

where $\mathbf{A}$ is a linear map given by an $N\times 3$ matrix.

We did mention in the introduction that $\boldsymbol{\alpha}$ are chosen to be frame indifferent. If they would remain invariant under all mappings of the isotropy group too then hypothesis $(1.11)_1$ will imply $\hat{\mathbf{q}}(\mathbf{F},\theta,\mathbf{0},\boldsymbol{\alpha})=\mathbf{0}$ for isotropic materials (see Truesdell and Noll[12], relation (96.22)). One therefore has to assume that at least certain components of $\boldsymbol{\alpha}$ do change under the isotropy group of the material.

The above constitutive assumptions lead to a quasilinear partial differential system for the unknown functions. For the one-dimensional case, these hypotheses have been laid down by Perzyna and Kosiński[2] and have been investigated by Suliciu[4, 5] in the frame of wave symmetry and of Cattaneo's hyperbolic heat conduction constitutive assumption[6, 7].

Inequality (1.8) and the constitutive assumptions (1.11) lead to

$$\begin{aligned}&\hat{\mathbf{q}}(\mathbf{F},\theta,\boldsymbol{\alpha})=-\rho_0\theta\mathbf{A}^T(\boldsymbol{F},\theta,\boldsymbol{\alpha})\frac{\partial\hat{\psi}}{\partial\boldsymbol{\alpha}}(\mathbf{F},\theta,\boldsymbol{\alpha})\\ &\frac{\partial\hat{\psi}}{\partial\boldsymbol{\alpha}}(\mathbf{F},\theta,\boldsymbol{\alpha})\cdot\mathbf{b}(\mathbf{F},\theta,\boldsymbol{\alpha})\leq 0.\end{aligned} \tag{1.12}$$

2. SYMMETRIC ACCELERATION WAVES

A regular surface $\varphi(\mathbf{X},t)=0$, with $\mathbf{X}\in\mathcal{R}$ and $t\in R$ will be called an acceleration wave if $\mathbf{v}=\dot{\mathbf{x}}$, $\mathbf{F}=\text{Grad}\,\boldsymbol{\chi}$, θ and $\boldsymbol{\alpha}$ are continuous across it but their derivatives with respect to $\mathbf{X}$ and t have jump discontinuities when crossing this surface. It will be assumed that $\mathbf{b}$ and r have no jumps across the wave.

The geometric and kinematic compatibility conditions that have to be satisfied by the derivatives of $\mathbf{v}$, $\mathbf{F}$, θ and $\boldsymbol{\alpha}$ are (the reader unfamiliar with kinematic and dynamic compatibility conditions is directed to Truesdell and Toupin[13] or, for their applications to different types of materials, to Coleman, Gurtin, Herrera and Truesdell [15])

$$\left[\frac{\partial^2\chi_k}{\partial t^2}\right] = U^2_{a_k}, \quad \left[\frac{\partial F_{kl}}{\partial X_j}\right] = a_k n_l n_j, \quad \left[\frac{\partial F_{ij}}{\partial t}\right] = -U a_i n_j \tag{2.1}$$

$$\left[\frac{\partial\theta}{\partial t}\right] = -U\nu, \quad \left[\frac{\partial\theta}{\partial X_j}\right] = \nu n_j \tag{2.2}$$

$$\left[\frac{\partial\alpha_i}{\partial t}\right] = -U\gamma_i, \quad \left[\frac{\partial\alpha_i}{\partial X_j}\right] = \gamma_i n_j. \tag{2.3}$$

Here, n_j are the components of the unit vector $\mathbf{n}$, normal to the discontinuity surface, for a fixed t, and called the direction of propagation, U is the speed of propagation of the discontinuity surface and one has

$$n_i = (\partial\varphi/\partial X_i)/|\text{Grad}\,\varphi|, \quad U = -(\partial\varphi/\partial t)/|\text{Grad}\,\varphi|. \tag{2.4}$$

a_i are the components of the vector $\mathbf{a}$, called the mechanical amplitude of the wave, the scalar ν is called the thermal amplitude of the wave and the N-dimensional vector $\boldsymbol{\gamma}$ is called the internal state amplitude of the wave.

According to $(1.10)_{2,3}$ and $(1.12)_1$ one can write the following jump conditions

$$[\text{Div}\,\tilde{\mathbf{S}}]_i = \rho_0\left\{\frac{\partial^2\hat{\psi}}{\partial F_{ij}\partial F_{kl}} a_k n_l n_j + \nu\frac{\partial^2\hat{\psi}}{\partial\theta\partial F_{ij}} n_j + \frac{\partial^2\hat{\psi}}{\partial F_{ij}\partial\alpha_k}\gamma_k n_j\right\} \tag{2.5}$$

$$\left[\frac{\partial\eta}{\partial t}\right] = U\frac{\partial^2\hat{\psi}}{\partial\theta\partial F_{ij}} a_i n_j + \nu U\frac{\partial^2\hat{\psi}}{\partial\theta^2} + U\frac{\partial^2\hat{\psi}}{\partial\theta\partial\alpha_k}\gamma_k \tag{2.6}$$

$$[\text{Div}\,\hat{\mathbf{q}}] = \frac{\partial\hat{q}_i}{\partial F_{kl}} a_k n_l n_i + \nu\frac{\partial\hat{q}_i}{\partial\theta} n_i + \frac{\partial\hat{q}_i}{\partial\alpha_k}\gamma_k n_i. \tag{2.7}$$

Using (2.5) and $(2.1)_1$ in the above obtained dynamic compatibility condition $(1.6)_1$, we get

$$U^2 a_i = \frac{\partial^2\hat{\psi}}{\partial F_{ij}\partial F_{kl}} a_k n_l n_j + \nu\frac{\partial^2\hat{\psi}}{\partial\theta\partial F_{ij}} n_j + \frac{\partial^2\hat{\psi}}{\partial F_{ij}\partial\alpha_k}\gamma_k n_j. \tag{2.8}$$

From (2.6), (2.7) and $(1.6)_3$ there results

$$\rho_0\theta U\frac{\partial^2\hat{\psi}}{\partial\theta\partial F_{ij}} a_i n_j + \frac{\partial\hat{q}_i}{\partial F_{kl}} a_k n_l n_i + \rho_0\theta\nu\frac{\partial^2\hat{\psi}}{\partial\theta^2} U + \nu\frac{\partial\hat{q}_i}{\partial\theta} n_i + \rho_0\theta U\frac{\partial^2\hat{\psi}}{\partial\theta\partial\alpha_k}\gamma_k - \rho_0 U\frac{\partial\hat{\psi}}{\partial\alpha_k}\gamma_k + \frac{\partial\hat{q}_i}{\partial\alpha_k} n_i\gamma_k = 0 \tag{2.9}$$

while from (2.3), $(1.9)_5$ and $(1.11)_2$ we find

$$U\gamma_k + \nu A_{kj} n_j = 0. \tag{2.10}$$

For a fixed state $(\mathbf{F}, \theta, \boldsymbol{\alpha})$ and a given direction of propagation $\mathbf{n}$, the system (2.8)–(2.10) represents a homogeneous linear system of $N+4$ equations with $N+4$ unknowns: $\mathbf{a}$, ν and $\boldsymbol{\gamma}$. It will have a non-trivial solution if and only if its determinant vanishes.

In order to simplify the writing one may introduce the following notations

$$Q_{ik} = \frac{\partial^2\hat{\psi}}{\partial F_{ij}\partial F_{kl}} n_j n_l, \quad v = -\rho_0\theta\frac{\partial^2\hat{\psi}}{\partial\theta^2} \tag{2.11}$$

$$P_i = \frac{\partial^2\hat{\psi}}{\partial\theta\partial F_{ij}} n_j \tag{2.12}$$

$$D_i = \frac{\partial^2 \hat{\psi}}{\partial F_{ij} \partial \alpha_k} A_{kp} n_j n_p, \quad E_i = \frac{\partial \hat{q}_k}{\partial F_{il}} n_k n_l \tag{2.13}$$

$$z = -\frac{\partial \hat{q}_l}{\partial \alpha_k} A_{kp} n_l n_p, \quad w = \rho_0 \frac{\partial \hat{\psi}}{\partial \alpha_k} A_{kp} n_p - \rho_0 \theta \frac{\partial^2 \hat{\psi}}{\partial \theta \partial \alpha_k} A_{kp} n_p + \frac{\partial \hat{q}_l}{\partial \theta} n_l \tag{2.14}$$

$$H_{ik} = \frac{\partial^2 \hat{\psi}}{\partial F_{ij} \partial \alpha_k} n_j, \quad G_k = \rho_0 \theta \frac{\partial^2 \hat{\psi}}{\partial \theta \partial \alpha_k} - \rho_0 \frac{\partial \hat{\psi}}{\partial \alpha_k} \tag{2.15}$$

$$K_k = \frac{\partial \hat{q}_i}{\partial \alpha_k} n_i, \quad L = \frac{\partial \hat{q}_i}{\partial \theta} n_i, \quad M_k = A_{kj} n_j. \tag{2.16}$$

The quantities $\mathbf{Q} = (Q_{ij})$ and v/ρ_0 are called the homothermal acoustical tensor and the heat capacity at constant strain and internal state variables, respectively.

By means of these notations the determinant of the system (2.8)–(2.10) is written as

$$\Delta = \begin{vmatrix} Q_{11} - U^2 & Q_{12} & Q_{13} & P_1 & H_{11} & \cdots & H_{1N} \\ Q_{21} & Q_{22} - U^2 & Q_{23} & P_2 & H_{21} & \cdots & H_{2N} \\ Q_{31} & Q_{32} & Q_{33} - U^2 & P_3 & H_{31} & \cdots & H_{3N} \\ \rho_0 \theta U P_1 + E_1 & \rho_0 \theta U P_2 + E_2 & \rho_0 \theta U P_3 + E_3 & -Uv + L & UG_1 + K_1 & \cdots & UG_N + K_N \\ 0 & 0 & 0 & M_1 & U & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & M_N & 0 & \cdots & U \end{vmatrix}$$

$$\begin{aligned} &= U^{N-1}\{-vU^8 + wU^7 + (vI_{\mathbf{Q}} + z + \rho_0 \theta \mathbf{P} \cdot \mathbf{P}) U^6 + [-wI_{\mathbf{Q}} + \mathbf{P} \cdot (\mathbf{E} - \rho_0 \theta \mathbf{D})] U^5 \\ &\quad - (vII_{\mathbf{Q}} + zI_{\mathbf{Q}} + \mathbf{D} \cdot \mathbf{E} + \rho_0 \theta I_{\mathbf{Q}} \mathbf{P} \cdot \mathbf{P} - \rho_0 \theta \mathbf{QP} \cdot \mathbf{P}) U^4 \\ &\quad + [wII_{\mathbf{Q}} + I_{\mathbf{Q}} \mathbf{P} \cdot (\rho_0 \theta \mathbf{D} - \mathbf{E}) + \mathbf{QP} \cdot (\mathbf{E} - \rho_0 \theta \mathbf{D})] U^3 \\ &\quad + [vIII_{\mathbf{Q}} + zII_{\mathbf{Q}} + I_{\mathbf{Q}} \mathbf{D} \cdot \mathbf{E} - \mathbf{QD} \cdot \mathbf{E} - \rho_0 \theta I_{\mathbf{Q}} \mathbf{QP} \cdot \mathbf{P} + \rho_0 \theta II_{\mathbf{Q}} \mathbf{P} \cdot \mathbf{P} + \rho_0 \theta \mathbf{Q}^2 \mathbf{P} \cdot \mathbf{P}] U^2 \\ &\quad + [-wIII_{\mathbf{Q}} + II_{\mathbf{Q}} \mathbf{P} \cdot (\mathbf{E} - \rho_0 \theta \mathbf{D}) + I_{\mathbf{Q}} \mathbf{QP} \cdot (\rho_0 \theta \mathbf{D} - \mathbf{E}) + \mathbf{Q}^2 \mathbf{P} \cdot (\mathbf{E} - \rho_0 \theta \mathbf{D})] U \\ &\quad - zIII_{\mathbf{Q}} + I_{\mathbf{Q}} \mathbf{QD} \cdot \mathbf{E} - II_{\mathbf{Q}} \mathbf{D} \cdot \mathbf{E} - (\mathbf{Q}^2 \mathbf{D}) \cdot \mathbf{E}\} = 0 \end{aligned} \tag{2.17}$$

where $I_{\mathbf{Q}}$, $II_{\mathbf{Q}}$, $III_{\mathbf{Q}}$ are the invariants of $\mathbf{Q}$.

Equation (2.17) that determines the speed of propagation of acceleration waves has $U = 0$ as root of order $N - 1$ and this is due to the constitutive assumptions $(1.9)_5$ and (1.11). Acceleration waves for which $U = 0$ are called stationary acceleration waves.

From the physical point of view one expects to get symmetric acceleration waves, and therefore symmetric roots for equation (2.17) (i.e. if $U = U_1$ is a root of equation (2.17) then $U = -U_1$ is a root too). If, in relation (2.17), which is a polynomial of degree $N + 7$ in U, the coefficient of U^{N-1} is different from zero, there will be only $N - 1$ roots that are equal to zero. The remaining eight roots are generally neither real nor symmetric. We now lay down the following constitutive assumption that leads to symmetric speeds of propagation.

Symmetry assumption I: *For any fixed state* $(\mathbf{F}, \theta, \boldsymbol{\alpha})$ *and any fixed direction of propagation* $\mathbf{n}$, *the equation that gives the speeds of propagation of acceleration waves must admit symmetric roots.*

Symmetry assumption I is satisfied if and only if the following conditions

$$\begin{aligned} w &= 0 \\ \mathbf{P} \cdot (\mathbf{E} - \rho_0 \theta \mathbf{D}) &= 0 \\ \mathbf{QP} \cdot (\mathbf{E} - \rho_0 \theta \mathbf{D}) &= 0 \\ \mathbf{Q}^2 \mathbf{P} \cdot (\mathbf{E} - \rho_0 \theta \mathbf{D}) &= 0 \end{aligned} \tag{2.18}$$

are verified for any state $(\mathbf{F}, \theta, \boldsymbol{\alpha})$ *and any direction of propagation* $\mathbf{n}$. Under these conditions and denoting $Z = U^2$, eqn (2.17) becomes

$$vZ^4-(vI_\mathbf{Q}+z+\rho_0\theta\mathbf{P}\cdot\mathbf{P})Z^3+\{vII_\mathbf{Q}+zI_\mathbf{Q}+\mathbf{D}\cdot\mathbf{E}+\rho_0\theta(I_\mathbf{Q}\mathbf{P}\cdot\mathbf{P}-\mathbf{QP}\cdot\mathbf{P})\}Z^2$$
$$-\{vIII_\mathbf{Q}+zII_\mathbf{Q}+I_\mathbf{Q}\mathbf{D}\cdot\mathbf{E}-\mathbf{QD}\cdot\mathbf{E}$$
$$+\rho_0\theta(II_\mathbf{Q}\mathbf{P}\cdot\mathbf{P}-I_\mathbf{Q}\mathbf{QP}\cdot\mathbf{P}+\mathbf{Q}^2\mathbf{P}\cdot\mathbf{P})\}Z+zIII_\mathbf{Q}-I_\mathbf{Q}\mathbf{QD}\cdot\mathbf{E}+II_\mathbf{Q}\mathbf{D}\cdot\mathbf{E}+\mathbf{Q}^2\mathbf{D}\cdot\mathbf{E}=0. \quad (2.19)$$

Conditions $(2.18)_{2-4}$ can also be written as

$$\begin{aligned}\mathbf{P}\cdot\mathbf{W}&=0\\ \mathbf{QP}\cdot\mathbf{W}=\mathbf{P}\cdot\mathbf{QW}&=0\\ \mathbf{Q}^2\mathbf{P}\cdot\mathbf{W}=\mathbf{QP}\cdot\mathbf{QW}=\mathbf{P}\cdot\mathbf{Q}^2\mathbf{W}&=0\end{aligned} \quad (2.20)$$

where $\mathbf{W}=\mathbf{E}-\rho_0\theta\mathbf{D}$. Therefore, *if* $\mathbf{P}$ *and* $\mathbf{W}$ *are both different from zero then either* $\mathbf{P}$ *or* $\mathbf{W}$ *is an eigenvector of* $\mathbf{Q}$. Let $(\mathbf{F}, \theta, \boldsymbol{\alpha})$ and $\mathbf{n}$ be fixed and let $U^2 \neq 0$ be a corresponding root of eqn (2.19). From (2.10) and (2.16) one has

$$\gamma_k = -\frac{\nu}{U}M_k. \quad (2.21)$$

Using the notations (2.11)–(2.16), from (2.9) together with $(2.18)_1$ and (2.21) one gets

$$(vU^2-z)\nu = U(U\rho_0\theta\mathbf{P}+\mathbf{E})\cdot\mathbf{a} \quad (2.22)$$

and from (2.8) and (2.21) there results

$$U^2\mathbf{a}-\mathbf{Qa}=\nu\left(\mathbf{P}-\frac{\mathbf{D}}{U}\right). \quad (2.23)$$

We make now a second constitutive assumption concerning symmetry.

Symmetry assumption II: *For a real acceleration wave that travels on direction* $\mathbf{n}$ *with the mechanical amplitude* $\mathbf{a}$, *U is a possible speed of propagation if and only if $-U$ is.*

This symmetry assumption is automatically satisfied in case of a non-linear elastic material (see Truesdell [14, 15], Section 2). We here require that the constitutive equations also verify such an hypothesis in case of a material with internal state variables.

Suppose now that eqn (2.19) has a real root $Z_1>0$; then, corresponding to Z_1, the homogeneous system (in $\mathbf{a}$ and ν) (2.22)–(2.23) has the real solutions $(\mathbf{a}, \nu)$ and $(\mathbf{a}', \nu')$ for $U_1=\sqrt{Z_1}$ and $-U_1=-\sqrt{Z_1}$ respectively. According to symmetry assumption II, if $(\mathbf{a}, \nu)$ is the solution that corresponds to $U=U_1$, $(\mathbf{a}, \bar{\nu})$ must be the solution that corresponds to $U=-U_1$. Therefore, from (2.22) and (2.23), there follows that

$$\begin{aligned}(U_1^2v-z)(\nu-\bar{\nu})&=2U_1\mathbf{E}\cdot\mathbf{a}\\ (\nu-\bar{\nu})\mathbf{P}&=\frac{1}{U_1}(\nu+\bar{\nu})\mathbf{D}.\end{aligned} \quad (2.24)$$

3. STRONG EQUILIBRIUM STATES. ELLIPTICITY CONDITION

(1) A state $(\mathbf{F}^*, \theta^*, \mathbf{g}^*=\mathbf{0}, \boldsymbol{\alpha}^*)$ is called an *equilibrium state* if

$$\mathbf{f}^*=\mathbf{f}(\mathbf{F}^*, \theta^*, \mathbf{0}, \boldsymbol{\alpha}^*)=\mathbf{0}. \quad (3.1)$$

If an equilibrium state satisfies the additional condition

$$\frac{\partial\psi^*}{\partial\alpha_k}=\frac{\partial\hat{\psi}}{\partial\alpha_k}(\mathbf{F}^*, \theta^*, \boldsymbol{\alpha}^*)=0 \quad (3.2)$$

it is said to be a *strong equilibrium state*. In the following, whenever we use a star as upper index on a quantity, it means that the quantity refers to a strong equilibrium state.

For a strong equilibriumm state, $(1.12)_1$ implies

$$\tilde{\mathbf{q}}^* = \tilde{\mathbf{q}}(\mathbf{F}^*, \theta^*, \mathbf{0}, \boldsymbol{\alpha}^*) = \mathbf{0} \tag{3.3}$$

$$\frac{\partial \tilde{\mathbf{q}}^*}{\partial \theta} + \rho_0 \theta^* \mathbf{A}^{*T} \frac{\partial^2 \psi^*}{\partial \boldsymbol{\alpha} \partial \theta} = \mathbf{0} \tag{3.4}$$

$$\frac{\partial \tilde{\mathbf{q}}^*}{\partial \mathbf{F}} + \rho_0 \theta^* \mathbf{A}^{*T} \frac{\partial^2 \psi^*}{\partial \boldsymbol{\alpha} \partial \mathbf{F}} = \mathbf{0} \tag{3.5}$$

$$\frac{\partial \tilde{\mathbf{q}}^*}{\partial \boldsymbol{\alpha}} + \rho_0 \theta^* \mathbf{A}^{*T} \frac{\partial^2 \psi^*}{\partial \boldsymbol{\alpha}^2} = \mathbf{0}. \tag{3.6}$$

For further implications of the Clausius–Duhem inequality in case of a strong equilibrium state see Bowen[9], Truesdell[8].

(2) The Clausius–Duhem inequality, interpreted in the sense of Coleman and Noll[16], imposes important restrictions on the form of the constitutive equations. For materials with internal state variables these restrictions have been established by Coleman and Gurtin[1] (see also Bowen[9]). However, these restrictions still allow the constitutive equations to remain too general, so they may include certain effects that are inadmissible from the physical point of view.

In order to remove such undesirable effects in nonlinear elasticity, in the literature there are proposed several so called adscititious inequalities (for a detailed discussion on this subject see Truesdell and Noll[12]).

In case of materials with internal state variables Bowen[9] suggests an inequality involving the function $\hat{\psi}$, an inequality of the same type as that of Coleman and Noll[17], and investigates the consequences on equilibrium states and on the second order derivatives of the function $\hat{\psi}$ with respect to its arguments. We here follow the same idea but impose on functions $\hat{\psi}$ and $\mathbf{f}$ an inequality of elliptic type. (See Hadamard[18], Truesdell[14], and also Truesdell and Noll[12].) In Section 4 will be seen the important consequences this inequality has for the propagation of waves in the neighborhood of a strong equilibrium state.

We require that functions $\hat{\psi}$ and $\mathbf{f}$ be defined such that the inequality

$$\hat{\psi}(\mathbf{F}, \theta, \boldsymbol{\alpha}) - \hat{\psi}(\tilde{\mathbf{F}}, \tilde{\theta}, \tilde{\boldsymbol{\alpha}}) - (\theta - \tilde{\theta}) \frac{\partial \hat{\psi}}{\partial \theta}(\mathbf{F}, \theta, \boldsymbol{\alpha}) - tr\left\{(\mathbf{F} - \tilde{\mathbf{F}})^T \frac{\partial \hat{\psi}}{\partial \mathbf{F}}(\tilde{\mathbf{F}}, \tilde{\theta}, \tilde{\boldsymbol{\alpha}})\right\} - (\boldsymbol{\alpha} - \tilde{\boldsymbol{\alpha}}) \frac{\partial \hat{\psi}}{\partial \boldsymbol{\alpha}}(\tilde{\mathbf{F}}, \tilde{\theta}, \tilde{\boldsymbol{\alpha}}) > 0 \tag{3.7}$$

holds for any $(\mathbf{F}, \theta, \boldsymbol{\alpha})$, $(\tilde{\mathbf{F}}, \tilde{\theta}, \tilde{\boldsymbol{\alpha}})$ *with* $(\mathbf{F}, \theta, \boldsymbol{\alpha}) \neq (\tilde{\mathbf{F}}, \tilde{\theta}, \tilde{\boldsymbol{\alpha}})$,

$$\mathbf{F} = \tilde{\mathbf{F}} + \mathbf{a} \otimes \mathbf{b}, \quad \theta = \lambda + \tilde{\theta}, \quad \boldsymbol{\alpha} = \tilde{\boldsymbol{\alpha}} + \mathbf{f}_{\mathbf{g}}(\tilde{\mathbf{F}}, \tilde{\theta}, \mathbf{0}, \tilde{\boldsymbol{\alpha}})\mathbf{c} \tag{3.8}$$

where $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *are vectors in* R^3 *and* $\lambda \in R$, *that satisfy*

$$\{\operatorname{sgn}(\det \tilde{\mathbf{F}})\} \det(\tilde{\mathbf{F}} + \mathbf{a} \otimes \mathbf{b}) > 0, \quad \lambda + \tilde{\theta} > 0. \tag{3.9}$$

Inequality (3.7) implies that

$$\frac{\partial^2 \hat{\psi}}{\partial F_{ij} \partial F_{kl}} a_i a_k b_j b_l + 2 \frac{\partial^2 \hat{\psi}}{\partial F_{ij} \partial \alpha_k} \frac{\partial f_k}{\partial g_l} a_i b_j c_l - \lambda^2 \frac{\partial^2 \hat{\psi}}{\partial \theta^2} + \frac{\partial^2 \hat{\psi}}{\partial \alpha_k \partial \alpha_l} \frac{\partial f_k}{\partial g_r} \frac{\partial f_l}{\partial g_s} c_r c_s \geq 0 \tag{3.10}$$

must be satisfied for any $\mathbf{a}, \mathbf{b}, \mathbf{c} \in R^3$ and $\lambda \in R$. *If* (3.10) *is a strict inequality, it is called the strong ellipticity condition.*

The strict inequality (3.10) *has the following immediate consequences*

$$\frac{\partial^2 \hat{\psi}}{\partial F_{ij} \partial F_{kl}} a_i a_k b_j b_l > 0 \tag{3.11}$$

$$-\frac{\partial^2\hat{\psi}}{\partial\theta^2}=\frac{\partial\hat{\eta}}{\partial\theta}>0 \tag{3.12}$$

$$\frac{\partial^2\hat{\psi}}{\partial\alpha_k\partial\alpha_l}\frac{\partial f_k}{\partial g_r}\frac{\partial f_l}{\partial g_s}c_rc_s>0 \tag{3.13}$$

for any $\mathbf{a}, \mathbf{b}, \mathbf{c}\in R^3$ *and any state* $(\mathbf{F}, \theta, \mathbf{g}=\mathbf{0}, \boldsymbol{\alpha})$.

According to (3.11), *the tensor* $\mathbf{Q}$ *given by* $(2.11)_1$ *will be positive definite while* (3.12) *is nothing else than*

$$v>0. \tag{3.14}$$

(3) We now present several results that hold for strong equilibrium states. *Under hypothesis* (1.13), *condition* (3.13) *together with* (3.6) *state that the quantity z defined by* $(2.14)_1$ *is positive at strong equilibrium states,* i.e.

$$z^*>0. \tag{3.15}$$

The symmetry condition $(2.18)_1$ written at a strong equilibrium state and (3.4) yield

$$\frac{\partial\hat{\mathbf{q}}^*}{\partial\theta}=\mathbf{0},\quad \mathbf{A}^{*T}\frac{\partial^2\hat{\psi}^*}{\partial\boldsymbol{\alpha}\partial\theta}=\mathbf{0}. \tag{3.16}$$

Then, from (3.5) and (2.13) one gets

$$\mathbf{E}^*=-\rho_0\theta^*\mathbf{D}^*, \tag{3.17}$$

while (3.17) and $(2.20)_1$ imply

$$\mathbf{P}^*\cdot\mathbf{D}^*=0 \tag{3.18}$$

so, if $\mathbf{P}^*\neq\mathbf{0}$ and $\mathbf{D}^*\neq\mathbf{0}$, by (2.20) there results that either $\mathbf{P}^*$ or $\mathbf{D}^*$ are eigenvectors for $\mathbf{Q}^*$.

4. REAL ACCELERATION WAVES

We will prove that *in the neighborhood of a strong equilibrium state, under the above stated symmetry and strong ellipticity hypotheses, all acceleration waves are real.*

We show first that, at strong equilibrium states, symmetry assumption I and the strong ellipticity hypothesis imply there exists a symmetric matrix $\mathbf{R}=(R_{ij})_{i,j=1,\dots,4}$ such that its characteristic equation

$$\det(\mathbf{R}-Z\mathbf{I})=0 \tag{4.1}$$

coincides with eqn (2.19). Hence all the roots of (2.19) will be real for any direction of propagation $\mathbf{n}$.

We choose

$$R_{kl}=\lambda_k^*\delta_{kl},\quad k,l=1,2,3 \tag{4.2}$$

where λ_k^*, $k=1,2,3$ are the eigenvalues of $\mathbf{Q}^*$. The remaining components $R_{i4}=R_{4i}$, $i=1,2,3,4$, will be determined in order to make eqn (4.1) coincide with eqn (2.19). One gets

$$R_{44}=\frac{1}{v^*}(z^*+\rho_0\theta^*\mathbf{P}^*\cdot\mathbf{P}^*) \tag{4.3}$$

and

$$\begin{aligned}
R_{14}^2+R_{24}^2+R_{34}^2&=\frac{\rho_0\theta^*}{v^*}(\mathbf{Q}^*\mathbf{P}^*\cdot\mathbf{P}^*+\mathbf{D}^*\cdot\mathbf{D}^*)\\
\lambda_1^*R_{14}^2+\lambda_2^*R_{24}^2+\lambda_3^*R_{34}^2&=\frac{\rho_0\theta^*}{v^*}(\mathbf{Q}^{*2}\mathbf{P}^*\cdot\mathbf{P}^*+\mathbf{Q}^*\mathbf{D}^*\cdot\mathbf{D}^*)\\
\lambda_1^{*2}R_{14}^2+\lambda_2^{*2}R_{24}^2+\lambda_3^{*2}R_{34}^2&=\frac{\rho_0\theta^*}{v^*}(\mathbf{Q}^{*3}\mathbf{P}^*\cdot\mathbf{P}^*+\mathbf{Q}^{*2}\mathbf{D}^*\cdot\mathbf{D}^*).
\end{aligned} \tag{4.4}$$

As there exists an orthonormal basis in R^3 consisting of the eigenvectors of $\mathbf{Q}^*$, then, by writing $\mathbf{Q}^*$, $\mathbf{P}^*$ and $\mathbf{D}^*$ in this new basis one immediately gets

$$R_{i4}^2 = \frac{\rho_0\theta^*}{v^*}(\lambda_i^* P_i^{*2} + D_i^{*2}), \quad i = 1,2,3 \tag{4.5}$$

where P_i^* and D_i^* are the components of $\mathbf{P}^*$ amd $\mathbf{D}^*$ respectively, with respect to the new basis.

Now let Z_i^*, $i = 1,2,3,4$ be the four real roots of eqn (2.19). From (3.14) and (3.15) one has

$$Z_1^* + Z_2^* + Z_3^* + Z_4^* = I_{\mathbf{Q}^*} + \frac{1}{v^*}(z^* + \rho_0\theta^*\mathbf{P}^* \cdot \mathbf{P}^*) > 0, \tag{4.6}$$

hence at least one of the Z_i^* is positive, say $Z_1^* = U_1^{*2} > 0$. Then, corresponding to the speed of propagation U_1^*, there exists a real solution $(\mathbf{a}_1^*, \nu_1^*, \gamma_1^*)$ of the homogeneous system (2.8)–(2.10).

Since in linear thermoelasticity $\mathbf{P}$ is directed along the normal $\mathbf{n}$ and its length is equal to the stress-temperature modulus divided by $\rho_0\theta^*$ (see Section 6), one may *assume that for any strong equilibrium state and any direction of propagation,*

$$\mathbf{P}^* \neq \mathbf{0}. \tag{4.7}$$

In linear thermoelasticity $\mathbf{P}$ is an eigenvector of $\mathbf{Q}$.

From (2.24) written for a strong equilibrium state and multiplied by $\mathbf{P}$, from (3.18) and (4.7) there follows

$$\nu_1^* = \tilde{\nu}_1^* \tag{4.8}$$

and

$$\nu_1^*\mathbf{D}^* = \mathbf{0}. \tag{4.9}$$

Equality (4.8) says that, *at strong equilibrium states, the strong ellipticity condition, the symmetry assumptions I and II and hypothesis* (4.7) *require the thermal amplitude to have the same symmetry property as the mechanical amplitude.*

From (2.23), (4.7) and (4.9) one can see that the thermal amplitude of the wave ν^* vanishes if and only if the mechanical amplitude $\mathbf{a}^*$ is an eigenvector of $\mathbf{Q}^*$.

According to (4.9), if there exists at least one non-zero thermal amplitude ν^*, one has $\mathbf{D}^* = \mathbf{0}$. Let us therefore investigate what happens when all ν^* vanish.

If $\nu_1^* = 0$, $\mathbf{a}_1^*$ will be an eigenvector of $\mathbf{Q}^*$ for the eigenvalue U_1^{*2}. Denoting by $\lambda_1^*, \lambda_2^*, \lambda_3^*$ the eigenvalues of $\mathbf{Q}^*$, one has $U_1^{*2} = \lambda_1^*$. Relation (4.6) will imply now the existence of another positive root for (2.19), say $Z_2^* > 0$. There are two cases: (a) $Z_2^* = Z_1^* = \lambda_1^*$, i.e. $Z = \lambda_1^*$ is a double root for eqn (2.19) but not a double eigenvalue for $\mathbf{Q}^*$ and (b) $Z_2^* \neq \lambda_1^*$ or $Z_2^* = \lambda_1^*$ but λ_1^* is a double eigenvalue of $\mathbf{Q}^*$.

For case (b), following again the above procedure, if $\nu_2^* = 0$, there results $Z_2^* = \lambda_2^*$ (λ_2^* being possibly equal to λ_1^*) and from (4.6) we get $Z_3^* > 0$. One faces again the two cases (a) and (b). Suppose we are in case (b): if $\nu_3^* = 0$ then $Z_3^* = \lambda_3^*$, hence $Z_4^* = (z^* + \rho_0\theta^*\mathbf{P}^* \cdot \mathbf{P}^*)/v^*$. But $Z_i^* = \lambda_i^*$, $i = 1,2,3$ and thus, according to eqn (2.19), one must have $\mathbf{D}^* \cdot \mathbf{D}^* + \mathbf{Q}^*\mathbf{P}^* \cdot \mathbf{P}^* = 0$ and therefore $\mathbf{D}^* = \mathbf{0}$ and $\mathbf{P}^* = \mathbf{0}$ which contradicts hypothesis (4.7).

In case (a), in order to simplify the calculations, let us write the system (2.22)–(2.23) in that orthogonal basis consisting only of the eigenvectors of $\mathbf{Q}^*$; since $Z = \lambda_1^*$ is a double root of eqn (2.19), for $U^2 = \lambda_1^*$ we get $P_1^* = 0$ (where P_i^* denote the components of $\mathbf{P}^*$ in the new basis) and the solutions of the system (2.22)–(2.23) will be

$$\left(a^*, \ \frac{P_2^*}{\lambda_1^* - \lambda_2^*}\nu^*, \ \frac{P_3^*}{\lambda_1^* - \lambda_3^*}\nu^*, \ \nu^*\right). \tag{4.10}$$

Hence, there exist such solutions for which $\nu^* \neq 0$.

If $U^2 = \lambda_1^*$ is a triple solution for eqn (2.19) but only a double eigenvalue of $\mathbf{Q}^*$, then

$P_1^* = P_2^* = 0$ and the solutions of the system (2.22)–(2.23) have the form

$$\left(a_1^*, \quad a_2^*, \quad \frac{\nu^* P_3^*}{\lambda_1^* - \lambda_3^*}, \quad \nu^*\right) \tag{4.11}$$

and the same conclusion as in the previous case follows.

Therefore, *if the hypotheses* (1.13), (4.7), *the symmetry assumptions I and II, and the strong ellipticity condition are satisfied then, for any direction of propagation* $\mathbf{n}$ *and any strong equilibrium state, eqn* (2.19) *can not have three roots that are eigenvalues of* $\mathbf{Q}^*$, *and*

$$\mathbf{D}^* = \mathbf{0}, \quad \mathbf{E}^* = \mathbf{0}. \tag{4.12}$$

Now, (3.11), (3.14), (3.15) and (4.12) imply that, *at strong equilibrium states, all four roots of eqn* (2.19) *are real and positive.* That is, *for any direction of propagation* $\mathbf{n}$, *in the neighborhood of any strong equilibrium state, there exist four real speeds of propagation* $U_i^* > 0$, $i = 1, \ldots, 4$ *and four amplitudes of the wave* $(\mathbf{a}_i^*, \nu_i^*)$, $i = 1, \ldots, 4$, *respectively. At least two of the four waves carry jumps of the temperature derivatives.*

According to (4.12), at a strong equilibrium state we may write the eqn (2.19) and the system (2.22)–(2.23) as follows

$$\begin{aligned} &v^* Z^4 - (v^* I_{\mathbf{Q}^*} + z^* + \rho_0 \theta^* \mathbf{P}^* \cdot \mathbf{P}^*) Z^3 + [v^* II_{\mathbf{Q}^*} + z^* I_{\mathbf{Q}^*} + \rho_0 \theta^* (I_{\mathbf{Q}^*} \mathbf{P}^* \cdot \mathbf{P}^* - \mathbf{Q}^* \mathbf{P}^* \cdot \mathbf{P}^*)] Z^2 \\ &\quad - [v^* III_{\mathbf{Q}^*} + z^* II_{\mathbf{Q}^*} + \rho_0 \theta^* (II_{\mathbf{Q}^*} \mathbf{P}^* \cdot \mathbf{P}^* - I_{\mathbf{Q}^*} \mathbf{Q}^* \mathbf{P}^* \cdot \mathbf{P}^* + \mathbf{Q}^{*2} \mathbf{P}^* \cdot \mathbf{P}^*)] Z + z^* III_{\mathbf{Q}^*} = 0 \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} (v^* U^{*2} - z^*) \nu^* &= \rho_0 \theta^* \mathbf{P}^* \cdot \mathbf{a}^* \\ U^{*2} \mathbf{a}^* - \mathbf{Q}^* \mathbf{a}^* &= \nu^* \mathbf{P}^*. \end{aligned} \tag{4.14}$$

5. HEAT FLUX IN THE NEIGHBORHOOD OF A STRONG EQUILIBRIUM STATE

Under assumption $(1.11)_1$, the principle of frame-indifference (see Coleman and Gurtin[1], relation $(12.4)_4$), implies that the response function that determines $\mathbf{q}$ can be written as

$$\mathbf{q} = \hat{\hat{\mathbf{q}}}(\mathbf{F}, \theta, \boldsymbol{\alpha}) = \mathbf{F} \bar{\mathbf{q}}^+(\mathbf{C}, \theta, \boldsymbol{\alpha}) \tag{5.1}$$

where $\mathbf{C} = \mathbf{F}^T \mathbf{F}$. Then $(1.7)_1$ yields

$$\tilde{\mathbf{q}} = \hat{\mathbf{q}}(\mathbf{F}, \theta, \boldsymbol{\alpha}) = J \bar{\mathbf{q}}^+(\mathbf{C}, \theta, \boldsymbol{\alpha}) = \check{\mathbf{q}}(\mathbf{C}, \theta, \boldsymbol{\alpha}) \tag{5.2}$$

i.e. the response function $\check{\mathbf{q}}$ depends on the deformation gradient only by means of the right Cauchy–Green strain tensor $\mathbf{C}$.

We will show in the following that *for strong equilibrium states*

$$\frac{\partial \hat{\mathbf{q}}^*}{\partial \mathbf{F}} = \mathbf{0}. \tag{5.3}$$

Indeed, from $(2.13)_2$ and $(4.12)_2$ there follows

$$\frac{\partial \hat{q}_k^*}{\partial F_{il}} + \frac{\partial \hat{q}_l^*}{\partial F_{ik}} = 0. \tag{5.4}$$

From (5.2) one has

$$\frac{\partial \hat{q}_k^*}{\partial F_{il}} = 2 \frac{\partial \check{q}_k^*}{\partial C_{jl}} F_{ij}^* \tag{5.5}$$

and using (5.5) in (5.4) we get

$$\frac{\partial \check{q}_k^*}{\partial C_{jl}} + \frac{\partial \check{q}_l^*}{\partial C_{jk}} = 0. \tag{5.6}$$

Since $C_{ij} = C_{ji}$, (5.6) yields

$$\frac{\partial \check{q}_i^*}{\partial C_{jk}} = 0, \quad i, j, k = 1, 2, 3 \tag{5.7}$$

and (5.3) follows immediately.

Relations (3.5) and (5.3) imply

$$\frac{\partial^2 \hat{\psi}^*}{\partial F_{ij} \partial \alpha_r} A^*_{rk} = 0, \quad i, j, k = 1, 2, 3. \tag{5.8}$$

Therefore, taking into account $(1.12)_1$, $(3.16)_1$ and (5.3) we may write in the neighborhood of a strong equilibrium state

$$\tilde{q}_i(\mathbf{F}, \theta, \boldsymbol{\alpha}) = \frac{\partial \hat{q}_i^*}{\partial \alpha_k}(\alpha_k - \alpha_k^*) + 0_i^{(2)}(|\mathbf{F} - \mathbf{F}^*| + |\theta - \theta^*| + |\boldsymbol{\alpha} - \boldsymbol{\alpha}^*|). \tag{5.9}$$

We thus obtained the following result: *in the neighborhood of a strong equilibrium state the heat flux can be approximated by a function depending on internal state variables only.*

Now, conversely, *if at a strong equilibrium state the partial derivatives of* $\hat{\mathbf{q}}$ *with respect to* $\mathbf{F}$ *and* θ *vanish then, for that strong equilibrium state, the symmetry hypotheses I and II are satisfied.* This follows immediately if one uses relations (3.4) and (3.5) in (2.22), (2.23) and (2.17).

6. THE LINEAR THEORY

As an application of the previous considerations, we discuss here the implications of our assumptions for the linear theory.

Suppose ($\mathbf{F} = \mathbf{I}$, $\theta = \theta_0$, $\mathbf{g} = \mathbf{0}$, $\boldsymbol{\alpha} = \boldsymbol{\alpha}^0$) is a strong equilibrium state and

$$\hat{\psi}^* = 0, \quad \eta^* = 0, \quad \mathbf{S}^* = \mathbf{0}. \tag{6.1}$$

Moreover, the material is assumed isotropic. Under an orthogonal mapping $\mathbf{H}$ of the isotropy group, the heat flux $\tilde{\mathbf{q}}$ defined by $(1.7)_1$ becomes $\mathbf{H}^T\tilde{\mathbf{q}}$. For purpose of illustration only we will *assume* here *the first three components of the vector* $\boldsymbol{\alpha}$ *of internal state variables to behave, under* $\mathbf{H}$, *like* $\tilde{\mathbf{q}}$, *while the other components remain invariant* i.e. *we assume the following decomposition of* $\boldsymbol{\alpha}$

$$\boldsymbol{\alpha} = (\beta_1, \beta_2, \beta_3, \alpha_4, \ldots, \alpha_N) = (\boldsymbol{\beta}, \bar{\boldsymbol{\alpha}}) \tag{6.2}$$

where

$$\boldsymbol{\beta} \to \mathbf{H}^T\boldsymbol{\beta}, \quad \bar{\boldsymbol{\alpha}} \to \bar{\boldsymbol{\alpha}}. \tag{6.3}$$

We introduce the symmetric strain tensor $\boldsymbol{\epsilon}$ defined as

$$2\epsilon_{ij} = F_{ij} + F_{ji} - 2\delta_{ij}. \tag{6.4}$$

According to (6.3), and to material frame indifference and isotropy, the free energy becomes an isotropic function of the tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ and the vector $\boldsymbol{\beta}$ (see for instance Truesdell and Noll[12], Section 11). Moreover, if

$$(\epsilon_{ij}\epsilon_{ij})^{1/2} \ll 1, \quad |\theta - \theta_0|/\theta_0 \ll 1, \quad |\boldsymbol{\alpha} - \boldsymbol{\alpha}^0| \ll 1,$$

the free energy $\hat{\psi}(\mathbf{F}, \theta, \boldsymbol{\alpha}) = \tilde{\psi}(\boldsymbol{\epsilon}, \theta, \boldsymbol{\alpha})$ can be written in the neighborhood of a strong equilibrium state as

$$\rho_0\tilde{\psi}(\boldsymbol{\epsilon}, \theta, \alpha) = \frac{1}{2}\Big\{\lambda\epsilon_{rr}^2 + 2\mu\epsilon_{ij}\epsilon_{ij} - \frac{2\kappa}{\theta_0}(\theta-\theta_0)\epsilon_{rr} - \frac{v}{\theta_0}(\theta-\theta_0)^2 - \frac{2}{\theta_0}(\theta-\theta_0)\Delta_k(\alpha_k - \alpha_k^0) + \\ + 2\epsilon_{rr}\Gamma_k(\alpha_k - \alpha_k^0) + \Lambda_{kl}(\alpha_k - \alpha_k^0)(\alpha_l - \alpha_l^0)\Big\} \tag{6.5}$$

where λ and μ are Lammé constants, κ/θ_0 is the stress-temperature modulus, v/θ_0 is the heat capacity and Γ_k, Δ_k and Λ_{kl} are defined as

$$\rho_0\frac{\partial^2\tilde{\psi}^*}{\partial F_{ij}\partial\alpha_k} = \delta_{ij}\Gamma_k \tag{6.6}$$

$$\rho_0\theta_0\frac{\partial^2\tilde{\psi}^*}{\partial\theta\partial\alpha_k} = -\Delta_k \tag{6.7}$$

$$\rho_0\frac{\partial^2\tilde{\psi}^*}{\partial\alpha_k\partial\alpha_l} = \Lambda_{kl}. \tag{6.8}$$

All constants in (6.5)–(6.8) are calculated at the strong equilibrium state ($\mathbf{F} = \mathbf{I}$, $\theta = \theta_0$, $\mathbf{g} = \mathbf{0}$, $\boldsymbol{\alpha} = \boldsymbol{\alpha}^0$).

From $(3.16)_2$ and (5.8) there follows

$$\Gamma_k A^*_{ki} = 0, \quad \Delta_k A^*_{ki} = 0, \quad i = 1, 2, 3 \tag{6.9}$$

respectively.

Since the free energy ψ is given as a function of $\boldsymbol{\epsilon} = (\epsilon_{ij})$, θ and $\boldsymbol{\alpha}$, relations $(1.10)_{2,3}$ will determine $\tilde{\mathbf{S}}$ and $\tilde{\eta}$:

$$\tilde{S}_{ij} = \lambda\epsilon_{rr}\delta_{ij} + 2\mu\epsilon_{ij} - \frac{\kappa}{\theta_0}(\theta-\theta_0)\delta_{ij} + \Gamma_k(\alpha_k - \alpha_k^0)\delta_{ij} \tag{6.10}$$

$$\rho_0\theta_0\tilde{\eta} = \kappa\epsilon_{rr} + v(\theta-\theta_0) + \Delta_k(\alpha_k - \alpha_k^0). \tag{6.11}$$

The internal energy has the following expression

$$\rho_0\tilde{e} = \rho_0\tilde{\psi} + \rho_0\theta\tilde{\eta} = \kappa\epsilon_{rr} + v(\theta-\theta_0) + \Delta_k(\alpha_k - \alpha_k^0) + \frac{1}{2}\Big\{\lambda\epsilon_{rr}^2 + 2\mu\epsilon_{ij}\epsilon_{ij} + \frac{v}{\theta_0}(\theta-\theta_0)^2 \\ + 2\epsilon_{rr}\Gamma_k(\alpha_k - \alpha_k^0) + \Lambda_{kl}(\alpha_k - \alpha_k^0)(\alpha_l - \alpha_l^0)\Big\}. \tag{6.12}$$

As we proved in the previous section, in the neighborhood of a strong equilibrium state the heat flux can be approximated by a linear function of the internal state variables (see (5.9))

$$\tilde{q}_i(\boldsymbol{\epsilon}, \theta, \boldsymbol{\alpha}) = \frac{\partial\tilde{q}_i^*}{\partial\alpha_k}(\alpha_k - \alpha_k^0). \tag{6.13}$$

Taking into account (6.3) and (6.13) $\tilde{\mathbf{q}}$ is an isotropic vector function of one vector only, therefore

$$\tilde{\mathbf{q}} = -\gamma(\boldsymbol{\beta} - \boldsymbol{\beta}^0) \\ \frac{\partial\tilde{q}_i^*}{\partial\alpha_k} = 0, \quad i = 1, 2, 3, \quad k = 4, \ldots, N \tag{6.14}$$

where γ is a constant.

By using decomposition (6.2)–(6.3) and hypothesis $(1.11)_2$ the evolution eqn $(1.9)_5$ can be decomposed as follows

$$\dot{\beta}_i = A_{ik}g_{0k} + b_i, \quad i = 1, 2, 3 \tag{6.15}$$

$$\dot{\bar{\alpha}}_j = \bar{A}_{jk}g_{0k} + \bar{b}_j, \quad j = 4, \ldots, N. \tag{6.16}$$

The vector valued functions $\mathbf{b} = (b_1, b_2, b_3)$ and $\mathbf{A}\mathbf{g}_0 = (A_{1k}g_{0k}, A_{2k}g_{0k}, A_{3k}g_{0k})$ are isotropic functions of a symmetric tensor and a vector (see Truesdell and Noll[12], Section 13). Linearization in the neighborhood of the strong equilibrium state as well as isotropy lead to the following form of eqn (6.15)

$$\dot{\beta}_i = Ag_{0i} + \delta(\beta_i - \beta_i^0) \tag{6.17}$$

where A and δ are constants.

Since we did assume $\bar{\alpha}_j$, $j = 4, \ldots, N$ to be scalars, the same argument as above leads to

$$A^*_{ik} = A\delta_{ik}, \quad \bar{A}^*_{jk} = 0, \quad i, k = 1, 2, 3, \quad j = 4, \ldots, N \tag{6.18}$$

while $\bar{b}_j, j = 4, \ldots, N$ do not depend on $\boldsymbol{\beta} - \boldsymbol{\beta}^0$ and are linear functions of the first invariant ϵ_{kk} of $\boldsymbol{\epsilon}$, of $\theta - \theta_0$ and $\bar{\alpha}_k - \alpha_k^0$, $k = 4, \ldots, N$.

According to $(6.14)_1$ the evolution eqn (6.17) becomes

$$-\frac{1}{\gamma}\dot{\bar{\mathbf{q}}} = \mathbf{A}\mathbf{g}_0 + \frac{\delta}{\gamma}\bar{\mathbf{q}}. \tag{6.19}$$

Equation (6.19) *is nothing else than Cattaneo's hyperbolic heat conduction equation* [6, 7]. *Thus, in the linear theory, the heat flux is an internal state variable and Cattaneo's equation represents an evolution equation for the heat flux.* Similar conclusions have been reached by Suliciu[4] in the one-dimensional case by using another hypothesis instead of the strong equilibrium state assumption. It is obvious that for slow processes eqn (6.19) can be approximated by Fourier's constitutive equation.

From (6.18) and (6.9) one gets

$$\Gamma_k = 0, \quad \Delta_k = 0, \quad k = 1, 2, 3 \tag{6.20}$$

and from $(2.14)_1$, (6.14) and (6.18) one has

$$z^* = \gamma A. \tag{6.21}$$

Now, let us write in this case the eqn (4.13) that gives the speeds of propagation. From (6.5), $(2.11)_1$ and (2.12) we obtain

$$\rho_0 Q_{ij} = (\lambda + \mu)n_i n_j + \mu\delta_{ij}, \quad P^*_i = -\frac{\kappa}{\rho_0\theta_0}n_i. \tag{6.22}$$

By using (6.22), eqn (4.13) becomes

$$(Z - a_t^2)^2\{Z^2 - (a_l^2 + a_c^2 + a_d^2)Z + a_c^2 a_l^2\} = 0 \tag{6.23}$$

where a_l and a_t are the speeds of propagation of the longitudinal and transversal mechanical acceleration waves respectively, a_c will be called the Cattaneo speed of propagation of thermal acceleration waves and a_d is a thermomechanical coupling coefficient. These quantities are defined by

$$a_l^2 = \frac{\lambda + 2\mu}{\rho_0}, \quad a_t^2 = \frac{\mu}{\rho_0}, \quad a_c^2 = \frac{z^*}{v^*}, \quad a_d^2 = \frac{\kappa^2}{\rho_0\theta_0 v^*}. \tag{6.24}$$

If $Z = a_t^2$ and $\mathbf{a}_t$ denotes the eigenvector of $\mathbf{Q}^*$ that corresponds to the eigenvalue a_t^2, then $\mathbf{a}_t \cdot \mathbf{n} = 0$ and, according to (4.12) and $(6.22)_2$ $(\mathbf{a}_t, \nu = 0)$ is a solution of the system (2.22)–(2.23). Thus, in the isotropic linear theory, for any direction of propagation $\mathbf{n}$, two transversal acceleration waves can exist, for which only the mechanical amplitude is non-zero, the same as for the purely mechanical case.

For $Z \neq a_t^2$ the system (2.22)–(2.23) that gives the wave amplitude $(\mathbf{a}, \nu)$ can be written, according to (4.12) and (6.22), as

$$(Z^* - a_c^2)\nu = -\frac{\kappa}{v^*} Z^*(\mathbf{a} \cdot \mathbf{n})$$

$$Z^* a_i - \left(\frac{\lambda + \mu}{\rho_0}(\mathbf{a} \cdot \mathbf{n})n_i \cdot + \frac{\mu}{\rho_0} a_i\right) = -\frac{\kappa}{\rho_0 \theta_0} \nu n_i.$$

There follows

$$\{Z^{*2} - (a_l^2 + a_c^2 + a_d^2)Z^* + + a_c^2 a_l^2\}(\mathbf{a} \cdot \mathbf{n}) = (Z^* - a_t^2)(\mathbf{a}_\perp \cdot \mathbf{a}_\perp) \tag{6.25}$$

where $\mathbf{a}_\perp$ is the component of $\mathbf{a}$ that is orthogonal to $\mathbf{n}$. Hence, if $Z^* = Z_1^* \neq a_t^2$ is a root of eqn (6.23), then $\mathbf{a}_\perp = \mathbf{0}$ and one has a longitudinal wave.

Therefore, *for any given direction* $\mathbf{n}$ *there always can be two transversal waves that propagate with the speed*

$$U^2 = \frac{\mu}{\rho_0} \tag{6.26}$$

which does not depend on direction $\mathbf{n}$, *and that carry no thermal jumps as* $\nu = 0$.

If

$$(\lambda + \mu)z^* \neq \mu\left((\lambda + \mu)v^* + \frac{\kappa}{\theta_0}\right), \tag{6.27}$$

i.e. μ/ρ_0 *is not a triple root for* (6.12), *then for any direction* $\mathbf{n}$ *there can always exist two longitudinal waves whose speeds of propagation are the roots of equation*

$$U^4 - (a_l^2 + a_c^2 + a_d^2)U^2 + a_c^2 a_l^2 = 0. \tag{6.28}$$

If $a_d \neq 0$ *or, equivalently, if condition* (4.7) *is satisfied, longitudinal acceleration waves will be coupled, i.e. both mechanical and thermal amplitudes,* $\mathbf{a}$ *and* ν *respectively, are different from zero.*

If (6.27) *does not hold, i.e.* $U^2 = \mu/\rho_0$ *is a root of eqn* (6.28) *then, as one can see from* (6.25) *and* (4.11), *the wave is generally neither longitudinal nor transversal.*

As we have shown in Section 3, inequality (3.13) implies $z^* > 0$. If one assumes that the left hand side of (3.13) vanishes for any $\mathbf{c} \in R^3$ then $z^* = 0$ for any $\mathbf{n} \in R^3$; the converse is also true. $z^* = 0$ will imply $a_c = 0$ and (6.28) has a root $U^2 = 0$ while the other one is given by

$$U_a^{*2} = a_l^2 + a_d^2. \tag{6.29}$$

U_a^* is sometimes called the adiabatic sound speed of propagation. Condition $z^* = 0$ will impose certain restriction on the way $\tilde{\mathbf{q}}$ and $\tilde{\psi}$ depend on $\boldsymbol{\alpha}$ but however, this does not imply that $\partial \hat{q}_l^* / \partial \alpha_k = 0$ and $\Lambda_{kl} = 0$.

Even for the nonlinear case, if $z^* = 0$ at a strong equilibrium state, from (2.22) and (2.23) one obtains

$$U^{*2}\mathbf{a} = \mathbf{Q}^*\mathbf{a} + \frac{\rho_0 \theta^*}{v^*}(\mathbf{P}^* \cdot \mathbf{a})\mathbf{P}^* \tag{6.30}$$

The tensor

$$\bar{\mathbf{Q}} = \mathbf{Q} + \frac{\rho_0 \theta}{v} \mathbf{P} \otimes \mathbf{P} \tag{6.31}$$

is called the adiabatic acoustic tensor.

Similar conclusions are reached in the theory of wave propagation in materials with fading memory (see Coleman and Gurtin[15], IV) if one assumes that the material is a non-conductor, i.e. $\tilde{\mathbf{q}} \equiv \mathbf{0}$; for materials with internal state variables see also Bowen and Wang[10] and Bowen and Chem[11].

7. SHOCK WAVES

(1) A smooth surface Σ of equation $\varphi(\mathbf{X}, t) = 0, \mathbf{X} \in \mathcal{R}$ (where $\mathcal{R}$ is the reference configuration of the body $\mathcal{B}$) is called a shock wave if the function $\mathbf{x} = \chi(\mathbf{X}, t)$ which describes the motion is continuous in $(\mathbf{X}, t)$ in the whole domain of definition but its derivatives $\mathbf{F} = \partial\chi/\partial\mathbf{X}, \mathbf{v} = \partial\chi/\partial t$ and the temperature θ have jumps across Σ, remaining continuous outside Σ.

If the surface Σ is regular at $(\mathbf{X}, t)$, i.e. Grad $\varphi \neq 0$ then, when crossing the surface at point $(\mathbf{X}, t)$ the following kinematic and dynamic compatibility conditions are satisfied (see Truesdell and Toupin[13], relations (189.1) as well as relations (205.3), (241.5) and (258.4) written in the initial reference configuration and using Piola–Kirchoff stress tensor).

$$[F_{ij}] = a_i n_j, \quad [v_i] = -Sa_i \tag{7.1}$$

$$\rho_0[v_i]S + [\tilde{S}_{ij}]n_j = 0 \tag{7.2}$$

$$\rho_0\left[e + \frac{v^2}{2}\right]S + [v_i\tilde{S}_{ij} - \tilde{q}_j]n_j = 0 \tag{7.3}$$

$$\rho_0 S[\eta] + \left[\frac{\tilde{q}_i}{\theta}\right]n_i \geq 0 \tag{7.4}$$

where $S = -(\partial\varphi/\partial t)/|\text{Grad}\,\varphi|$ is called the speed of propagation of the shock wave, $\mathbf{n}$ ($n_i = (\partial\varphi/\partial X_i)/|\text{Grad}\,\varphi|$) is the normal to Σ in $\mathcal{R}$ for a fixed t and $\mathbf{a}$ is the mechanical amplitude of the wave.

The evolution eqn $(1.9)_5$ *determines* $\boldsymbol{\alpha}(t)$, *for a fixed initial condition and a fixed* $\mathbf{X} \in \mathcal{R}$, *as a continuous function of t, for any* $\mathbf{F}(t)$, $\theta(t)$, $\mathbf{g}(t)$, *regulated functions of t*, $t \geq t_0$ (i.e. functions that for any t have left and right-side limits).

Now, applying a similar result to Proposition 2.5 of Suliciu[19] there follows that $[\boldsymbol{\alpha}] = 0$ across Σ if $S \neq 0$. *The jumps of all quantities* $\tilde{\mathbf{S}}$, e, η *and* $\tilde{\mathbf{q}}$ *are therefore determined only by the jumps of* $\mathbf{F}$, θ *and* $\mathbf{g}$, *if* $S \neq 0$.

The definition of an acceleration wave we have given at the beginning of Section 2 assumes $\boldsymbol{\alpha}$ continuous. The same argument as that used above shows that $\boldsymbol{\alpha}$ is continuous across any discontinuity surface for which $U \neq 0$. Therefore, in the definition of an acceleration wave, it is necessary to assume only the continuity of $\boldsymbol{\alpha}$ across stationary waves, i.e. those discontinuity surfaces for which $U = 0$.

From (7.1)–(7.3), by elimination of $[\mathbf{v}]$ and $\mathbf{a}$, one gets

$$S\left\{\rho_0[e] - \frac{1}{2}(\tilde{S}_{ij}^+ + \tilde{S}_{ij}^-)[F_{ik}]n_j n_k\right\} - [\tilde{q}_j]n_j = 0 \tag{7.5}$$

and

$$\rho_0^2 S^4 = \frac{[\tilde{S}_{ij}][\tilde{S}_{ik}]n_j n_k}{[F_{ij}][F_{ik}]n_j n_k}. \tag{7.6}$$

Now, if the normal $\mathbf{n}$ and the values $\mathbf{F}^+$, θ^+ and $\boldsymbol{\alpha}$ in front of the shock wave are fixed and hypothesis (1.11) is verified then, between the values $\mathbf{F}^-$ and θ^- beyond the shock wave there exists a relation determined by (7.5), (7.6) and the constitutive eqns (1.10) and $(1.12)_1$. In the one-dimensional case this type of relation is called the Hugoniot relation.

(2) *In the linear case, since the internal state variables do not jump across disconstinuity surfaces with* $S \neq 0$, *from* (6.14) *there follows*

$$[\tilde{q}_i] = 0. \tag{7.7}$$

Using (6.10)–(6.12) to express $\tilde{S}_{ij}$, $\tilde{\eta}$ and $\tilde{e}$ and the jump relations $(7.1)_1$ and (7.7), from (7.5) one obtains a relation between the mechanical amplitude and the thermal amplitude.

$$v[\theta] + \kappa(\mathbf{a} \cdot \mathbf{n}) = 0. \tag{7.8}$$

Together with (6.11), (7.8) implies

$$[\tilde{\eta}] = 0. \tag{7.9}$$

Therefore, *in case of the linear theory with internal state variables* as in usual linear thermoelasticity (under the assumption that the material is non-conductor) *all shocks are both adiabatic and isentropic.*

From (7.1), (7.2), (6.10) and (7.8) one obtains

$$\left\{\left(\lambda + \mu + \frac{\kappa^2}{\theta_0 v}\right) n_i n_j + (\mu - \rho_0 S^2)\delta_{ij}\right\} a_j = 0. \tag{7.10}$$

Equation (7.10) that gives the mechanical amplitude of the shock wave in the fixed direction of propagation $\mathbf{n}$ has non-trivial solutions only for those S^2 which satisfy the equation

$$(S^2 - a_t^2)^2(S^2 - a_l^2 - a_d^2) = 0. \tag{7.11}$$

Thus, we reached the following conclusions: *there can exist two types of shock waves, the same as for a non-conductor material: a double transversal wave across which temperature has no jump and a longitudinal wave that propagates with the adiabatic sound speed* (6.29) *and across which both mechanical and thermal amplitudes are different from zero.*

If one denotes by U_l^{*2} and U_h^{*2}, $U_l^{*2} < U_h^{*2}$ the two roots of eqn (6.28) then *the following inequality*

$$U_l^{*2} < a_l^2 < U_a^{*2} < U_h^{*2} \tag{7.12}$$

holds. This inequality that relates the propagation speeds of longitudinal shock and acceleration waves has the same form as the inequality obtained by Suliciu [4] for the one-dimensional case. He has shown that in case of impacting an undeformed elastic bar at rest, at room temperature θ_0, with a bar that moves against it with velocity V_0 and has temperature $\tilde{\theta}_0 \neq \theta_0$ and strain $\epsilon_0 \neq 0$, shock and acceleration waves will propagate in both bars. *The first one to propagate will be an acceleration wave that has propagation speed U_h^*, then a shock wave of speed U_a^* followed by an acceleration wave of speed U_l^*.*

The above picture describing time and space variation of thermomechanical quantities is similar to that given by usual linear thermoelasticity when for the heat flux one adopts a law of Fourier type (see Danilovskaia [20] and also Boley and Weiner [21], Chap. I).

The essential difference between the two cases lies in the fact that here the perturbation in front of the shock wave propagates with a finite speed.

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